International Journal of Theoretical Physics, Vol. 43, No. 10, October 2004 (© 2004)

Quantum Bouncer With Dissipation

G. López[1,2] and G. González[1]

Effects on the spectra of the quantum bouncer due to dissipation are given when a linear of quadratic dissipation is taken into account. Classical constants of motions and Hamiltonians are deduced for these systems and their quantized eigenvalues are estimated through perturbation theory. Differences were found comparing the eigenvalues of these two quantities.

KEY WORDS: dissipation; quantum bouncer.

1. INTRODUCTION

Dissipative systems has been one of the must subtle and difficult topics to deal with in classical (Dodonov, 1981; Glauber and Man'ko, 1984; López *et al.*, 1997) and quantum physics (López *et al.*, 2001; Okubo, 1981; Razavy, 1972). In general, to construct a consistent Lagrangian and Hamiltonian formulation for a given dissipative system can be a big challenge (López, 1996; López and González, 2003; Mijatovic *et al.*, 1984). There are basically two approaches to study dissipative systems. The first one tries to bring about the dissipation as a results of averaging over all the coordinates of the bath system, where one considers the whole system as composed of two parts, our original conservative system and the bath system which interacts with the conservative system and causes the dissipation (of energy) on it (Berman *et al.*, 2003; Caldeira and Leggett, 1983; Hu *et al.*, 1992; Unruh and Zurek, 1989). This approach has its own value and will not be followed or discussed here. The second approach considers that the bath system produces to our initially conservative system an average effect which is expressed as an additional external velocity depending force acting on the conservative system and transforming it into a dissipative system with this velocity depending force, the resulting classical dissipative system contains then this phenomenological (or theoretical) velocity depending force. Then, the question arises

[1] Departamento de Física de la Universidad de Guadalajara, Guadalajara, Jalisco, México.
[2] To whom correspondence should be addressed at Departamento de Física de la Universidad de Guadalajara, Apartado Postal 4-137, 44410 Guadalajara, Jalisco, México; e-mail: gulopez@udgserv.cencar.udg.mx.

over its consistent Lagrangian and Hamiltonian formalism and the consequences of its quantization. This approach, in addition, allows us to study and test the Hamiltonian approach for quantum mechanics and its consistence (López, 2002) and this is the approach we will follow in this paper. A system that has attracted our attention is the quantum bouncer. The quantum bouncer (Gean-Banacloche, 1999; Goodmanson, 2000) is the quantization of the motion of a particle which is attracted by the constant gravity force, that is, close to surface of the earth. This particle hits a perfectly reflexing surface, producing the bouncing effect. This system with an additional dissipation force has particular importance because of its potential experimental realization. This dissipative system has been studied very little so far, by using the first approach mentioned above (Onofrio and Viola, 1996).

We will assume that the external velocity depending force has linear and quadratic dependence with respect to the velocity. This approach gives us the opportunity to check the nature of quantization, using the Hamiltonian or constant of motion associated to the system, that is, by using the usual quantization of the linear-generalized momentum or using the quantization of the velocity. This consideration is particularly interesting in dissipative systems since one cannot always find a Hamiltonian as a function of the variables position and linear momentum (López, 1999a,b), that is, the velocity "v" cannot always be known explicitly in terms of the linear momentum "p" and position "x" of the particle through the relation $p = \partial L/\partial v$, where L is the Lagrangian of the system. This paper is organized as follows: we present the classical study for the dissipative system considering the linear and quadratic velocity depending force. The constant of motion, the Lagrangian, and the Hamiltonian of the system are derived, and we give their expressions up to second order in the dissipation parameter. We present the modification for the eigenvalues of the quantum bouncer, when this dissipation is taken into account, for the above-approximated (weak dissipation) constant of motion and Hamiltonian, using quantum perturbation theory. Finally, we present the conclusions and discussions of our results.

2. CLASSICAL LINEAR DISSIPATION

The motion of a particle of mass m under a constant gravitational force and a linear dissipative force is described by the equation

$$m\frac{d^2x}{dt^2} = -mg - \alpha v, \tag{1}$$

where x is the position of the particle, g is the constant acceleration due to earth gravity, α is the parameter which characterizes the dissipation, and $v = dx/dt$ is the velocity of the particle. A constant of motion of the autonomous system (1) is a function $K_\alpha = K_\alpha(x, v)$ satisfying the equation (López, 1999a,b)

$$v\frac{\partial K_\alpha}{\partial x} - \left(g + \frac{\alpha}{m}v\right)\frac{\partial K_\alpha}{v} = 0. \tag{2}$$

The solution of this equation such that $\lim_{\alpha\to 0} K_\alpha = mv^2/2 + mgx$ (the usual total energy for the nondissipative system) is given by

$$K_\alpha = \frac{m^2 gv}{\alpha} - m\left(\frac{mg}{\alpha}\right)^2 \ln\left(1 + \frac{\alpha v}{mg}\right) + mgx. \tag{3}$$

The Lagrangian associated to (1) can be obtained using the known expression (López, 2002)

$$L_\alpha = v\int \frac{K_\alpha(x, v)}{v^2}\,dv, \tag{4}$$

bringing about the following Lagrangian

$$L_\alpha = \frac{m^2 gv}{\alpha}\ln\left(1 + \frac{\alpha v}{mg}\right) + m\left(\frac{mg}{\alpha}\right)^2 \ln\left(1 + \frac{\alpha v}{mg}\right) - mgx - \frac{m^2 gv}{\alpha}. \tag{5}$$

Therefore, the generalized linear momentum and Hamiltonian are given by

$$p_\alpha = \frac{m^2 g}{\alpha}\ln\left(1 + \frac{\alpha v}{mg}\right) \tag{6}$$

and

$$H_\alpha = m\left(\frac{mg}{\alpha}\right)^2\left(\exp\left(\frac{\alpha p\alpha}{m^2 g}\right) - 1\right) - \frac{mg}{\alpha}p_\alpha + mgx. \tag{7}$$

At two orders in the dissipation parameter α, one has the constant of motion, the Lagrangian, the generalized linear momentum, and Hamiltonian given as

$$K = \frac{1}{2}mv^2 + mgx - \frac{\alpha}{3g}v^3 + \frac{\alpha^2}{4mg^2}v^4, \tag{8}$$

$$L = \frac{1}{2}mv^2 - mgx - \frac{\alpha}{6g}v^3 + \frac{\alpha^2}{12mg^2}v^4, \tag{9}$$

$$p = mv - \frac{\alpha}{2g}v^2 + \frac{\alpha^2}{3mg}v^3, \tag{10}$$

and

$$H = \frac{p^2}{2m} + mgx + \frac{\alpha}{6mg}p^3 + \frac{\alpha^2}{24m^5g^2}p^4. \tag{11}$$

The constant of motion (3) or (8a) and the Hamiltonian (7) or (8d) bring about the damping bouncing effect on the spaces (x, v) and (x, p). The dissipative parameter α can be determined by measuring the velocity v_o at the reflexing surface $(x = 0)$ and then measuring its maximum displacement $x_{\max}$ $(v = 0)$. Equaling the value of the constant of motion on both situations, one gets the expression

$$\frac{m^2gv_o}{\alpha} - m\left(\frac{mg}{\alpha}\right)^2 \ln\left(1 + \frac{\alpha v_o}{mg}\right) = mgx_{\max}, \tag{12}$$

where the parameter α can be obtain.

3. CLASSICAL QUADRATIC DISSIPATION

In this case, the motion of the particle is described by the equation

$$m\frac{d^2x}{dt^2} = -mg - \gamma v|v|, \tag{13}$$

where γ represents a dissipation constant which, of course, is different from the previous case. Proceeding in the same way as we did for the linear case, the constant of motion, Lagrangian, generalized linear momentum, and Hamiltonian are given by

$$K_{\pm} = \frac{1}{2}mv^2 \exp\left(\pm\frac{2\gamma x}{m}\right) \pm \frac{m^2g}{2\gamma}\left(\exp\left(\pm\frac{2\gamma x}{m}\right) - 1\right), \tag{14}$$

$$L_{\pm} = \frac{1}{2}mv^2 \exp\left(\pm\frac{2\gamma x}{m}\right) \pm \frac{m^2g}{2\gamma}\left(\exp\left(\pm\frac{2\gamma x}{m}\right) - 1\right), \tag{15}$$

$$p\pm = mv \exp\left(\pm\frac{2\gamma x}{m}\right), \tag{16}$$

and

$$H_{\pm} = \frac{p_{\pm}^2}{2m} \exp\left(\mp\frac{2\gamma x}{m}\right) \pm \frac{m^2g}{2\gamma}\left(\exp\left(\pm\frac{2\gamma x}{m}\right) - 1\right), \tag{17}$$

where the upper sign corresponds to the case $v \geq 0$, and the lower sign corresponds to the case $v < 0$. These equation where already given in reference (Negro and Tartaglia, 1980a,b). The damping effect of the bouncing particle in the space (x, v) can be traced in the following way: starting with the initial condition $x_o = 0$ and $v_o > 0$, for example, the constant of motion K_+ is determined,

$K_+ = mv_o^2/2$. Then, the maximum distance $x_{\max}(v = 0)$ is calculated from the expression $K_+ = (m^2 g/2\gamma)(\exp(2\gamma x_{\max}/m) - 1)$ which helps to calculate the constant K_-, $K_- = -(m^2 g/2\gamma)(\exp(-2\gamma x_{\max}/m) - 1)$. This K_- is used now to calculated the velocity at the turning point $(x = 0)$, $v_1^* = -\sqrt{2K_-/m}$. Considering a perfectly reflexing surface, the velocity of the bouncing particle for the next cycle is $v_1 = -v_1^*(v_1 < v_o)$, and the above cycle is reproduced again, and so on. Starting with the same initial conditions, the trajectories in this space are one below the other at any time, as the damping factor is grater. The damping effect in the space (x, p) through the Hamiltonian approach can be analyzed similarly. However, the trajectories starting with the same initial conditions on this space are not below the other all the time, as the damping factor is greater. This strange effect is due to the change in sign in (11d) with respect to (11a), produced by the position and velocity dependence of the expression (11c).

To determine the constant γ through the constant of motion, one can start with the initial conditions $(x_o = 0, v_o > 0)$ and can determined the constant of motion $K_+ = mv_o^2/2$. Then, one can measure the maximum displacement $x_{\max}(v = 0)$ and to solve γ from the equation

$$\frac{1}{2}mv_o^2 = \frac{m^2 g}{2\gamma}\left(\exp\left(\frac{2\gamma x_{\max}}{m}\right) - 1\right) \tag{18}$$

Up to second order in the dissipation parameter, one has from (11a) to (11d) the constant of motion, the Lagrangian, the generalized linear momentum, and the Hamiltonian given by

$$K_\pm = \frac{1}{2}mv^2 + mgx \pm \gamma[v^2 x + gx^2] + \gamma^2[v^2 x^2/m + 2gx^3/3m], \tag{19}$$

$$L_\pm = \frac{1}{2}mv^2 - mgx \pm \gamma[v^2 x + gx^2] + \gamma^2[v^2 x^2/m - 2gx^3/3m], \tag{20}$$

$$p\pm = mv \pm \gamma[2vx] + \gamma^2[2vx^2/m], \tag{21}$$

$$H_\pm = \frac{p^2}{2m} + mgx \mp \gamma[p^2 x/m^2 - gx^2] + \gamma^2[p^2 x^2/m^3 + 2gx^3/3m]. \tag{22}$$

4. QUANTIZATION OF THE CONSTANT OF MOTION

Equations (8a) and (13a) can be written as

$$K(x, v) = K_o(x, v) + V(x, v), \tag{23}$$

where K_o is the constant of motion without dissipation

$$K_o(x, v) = \frac{1}{2}mv^2 + mgx, \tag{24}$$

and V takes into account the dissipation factors

$$V(x, v) = \begin{cases} -\alpha\left(\frac{v^3}{3g}\right) + \alpha^2\left(\frac{v^4}{4mg^2}\right) & \text{(linear case)} \\ \mp\gamma[v^2x + gx^2] + \gamma^2\left[\frac{v^2x^2}{m} + \frac{2gm^3}{3m}\right] & \text{(quadratic case)} \end{cases} \tag{25}$$

The quantization of (14) can be carried out through the associated SchröUdinger's equation of this constant of motion

$$i\hbar\frac{\partial\Psi}{\partial t} = \widehat{K}(\widehat{x}, \widehat{v})\Psi, \tag{26}$$

where $\Psi = \Psi(x, t)$ is the wave function, $\hbar$ is the Plank constant divided by 2π, $\widehat{K} = \widehat{K}_o + \widehat{V}$ is a Hermitian operator associated to (17), and $\widehat{v}$ is the velocity operator defined as

$$\widehat{v} = -\frac{i\hbar}{m}\frac{\partial}{\partial x}. \tag{27}$$

Since Eq. (16) represents an stationary problem, the usual proposition $\Psi(x, t) = \exp(-iE^K t/\hbar)\psi(x)$ transforms (16) to an eigenvalue problem

$$(\widehat{K}_0 + \widehat{V})\psi = E^K\psi. \tag{28}$$

Taking the operator $\widehat{V}$ as a perturbation of the constant of motion K_o, one can calculate an approximated solution to the problem (18) through perturbation theory. The solution of the eigenvalue problem

$$\widehat{K}_o\psi_n^{(0)} = E_n^{(0)}\psi_n^{(0)} \tag{29}$$

is well known (Gean-Banacloche, 1999; Goodmanson, 2000), with $\psi_0^{(0)}$ being the eigenfunction given by

$$\psi_n^{(0)} = \frac{Ai(z - z_n)}{|Ai'(-z_n)|}, \tag{30}$$

where Ai and Ai' are the Airy function and its first differentiation, and z_n is its nth-zero ($Ai(-Z_n) = 0$) which ocurres for negative argument only. z is the normalized variable $z = x/l_g$ with $l_g = (\hbar^2/2m^2g)^{1/3}$, and z_n is related to the eigenvalue $E_n^{(0)}$ through the expression

$$z_n = \frac{E_n^{(0)}}{mgl_g}. \tag{31}$$

Up to second order in perturbation theory, the eigenvalues of (18) are given (in Dirac notation; Dirac, 1992) as

$$E_n^K = E_n^{(0)} + \langle n|\widehat{V}|n\rangle + \sum_{k\neq n}\frac{|\langle n|\widehat{V}|k\rangle|^2}{E_k^{(0)} - E_n^{(0)}}, \tag{32}$$

where $\langle z|n\rangle = \psi_n^{(0)}$. Using the Hermitian operators $\widehat{v^2x} = (\widehat{v}^2x + \widehat{v}x\widehat{v} + x\widehat{v}^2)/3$ and $\widehat{v^2x^2} = (\widehat{v}^2x^2 + \widehat{v}x^2\widehat{v} + x^2\widehat{v}^2 + x\widehat{v}^2x + x\widehat{v}x\widehat{v} + \widehat{v}x\widehat{v}x)/6$ for the associated expressions on (25b), and using the relations $\langle n|x^s|k\rangle = l_g^s\langle n|z^s|k\rangle$ and $\langle n|d^s/dx^s|k\rangle = l_g^{-s}\langle n|d^s/dz^s|k\rangle$ for any integer s, one has (see Appendix for a list of matrix elements)

$$E_n^K = E_n^{(0)} + \begin{cases} \alpha^2\left[\frac{l_g^2z_n^2}{5m} + \frac{8}{9}gl_g^3\sum_{k\neq n}\frac{|1/2+mgl_g/(E_k^{(0)}-E_n^{(0)})|^2}{E_k^{(0)}-E_n^{(0)}}\right] & \text{(linear)} \\ \mp\gamma\frac{12gl_g^2z_n^2}{15} + \gamma^2\left[\left(-\frac{1}{2}+\frac{56z_n^3}{105}\right)\frac{2gl_g^3}{m} + 4g^2l_g^4\sum_{k\neq n}a_{nk}\right], & \text{(quadratic)} \end{cases} \tag{33}$$

where a_{nk} is a real number given by

$$a_{nk} = \frac{|12 - 2z_k(z_n - z_k)^2 + (z_n - z_k)^3|2}{(z_k - z_n)^9}. \tag{34}$$

Note that for the linear dissipative case, there is no real contribution at first approximation, and for the quadratic dissipative case, the first-order contribution depends on whether the particle is moving up (-) or down (-). Within a full cycle, this first order correction is canceled out and the second-order contribution remains. Of course, for the approximation (23a) to be valid, one must have that the second term of this expression must be much lesser than $E_n^{(0)}$ which makes a restriction on the possible value of the dissipative parameter.

5. QUANTIZATION OF THE HAMILTONIAN

Equations (8d) and (13d) can be written as

$$H(x, p) = H_o(x, p) + W(x, p), \tag{35}$$

where H_o is the Haniltonian without dissipation,

$$H_o(x, p) = \frac{p^2}{2m} + mgx, \tag{36}$$

and W has the dissipation terms,

$$W(x, p) = \begin{cases} \alpha\left(\frac{p^3}{6mg}\right) + \alpha^2\left(\frac{p^4}{24m^5g^2}\right) & \text{(linear)} \\ \mp\gamma\left[\frac{p^2x}{m^2} - gx^2\right] + \gamma^2\left[\frac{p^2x^2}{m^3} + \frac{2gx^3}{3m}\right] & \text{(quadratic)} \end{cases} \tag{37}$$

It is necessary to mention that the quantization of some systems for quadratic dissipation has been solved by different authors (Borges *et al.*, 1988; Huang *et al.*, 1989; Mijatovic *et al.*, 1984; Negro and Tartaglia, 1980a,b; Razavy, 1983; Stuckeno and Kobe, 1986) but perfectly reflexing wall potential,

$$\tilde{V}(x) = \begin{cases} \infty & \text{for } x < 0 \\ mgx & \text{for } x \leq 0 \end{cases} \tag{38}$$

Moreover, the solution given in reference (Negro and Tartaglia, 1980a,b) is singular when the dissipation parameter goes to zero. Therefore, we think it worths to make the analysis of the quantization for small orders in the parameter γ. For the usual Shrödinger quantization approach, one has the stationary equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \widehat{H}(x, \widehat{p})\Psi, \tag{39}$$

where $\widehat{H}$ is the Hamiltonian operator associated to (24), and $\widehat{p}$ is the usual linear momentum operator $\widehat{p} = -i\hbar\partial/\partial x$. Equation (27) is transformed to an eigenvalue problem, $\widehat{H}\psi(x) = E^H\psi(x)$, through the proposition $\Psi(x,t) = \exp(-iE^H t/\hbar)\psi(x)$. Since the Hamiltonian $\widehat{H}$ is given by $\widehat{H} = \widehat{H}_o + \widehat{W}$, where the solution of the equation

$$\widehat{H}_o \psi_n^{(0)} = E_n^{(0)} \psi_n^{(0)} \tag{40}$$

is given by (20) and (21). Perturbation theory can be used to determine the approximated values of the eigenvalues E_n^H (similarly as done with expression (22)). Using the Hermitian operators $\widehat{p^2x} = (\widehat{p}^2x + \widehat{p}x\widehat{p} + x\widehat{p}^2)/3$ and $\widehat{p^2x^2} = (\widehat{p}^2x^2 + \widehat{p}x^2\widehat{V} + x^2\widehat{p}^2 + x\widehat{p}^2x + x\widehat{p}x\widehat{p} + \widehat{p}x\widehat{p}x)/6$ for the associated expressions on (25b), one gets

$$E_n^H = E_n^{(0)} + \begin{cases} \alpha^2 \left[\frac{l_g^2 z_n^2}{30m} + \frac{4}{9} g l_g^3 \sum_{k \neq n} \frac{|1/2 + mgl_g/(E_k^{(0)} - E_n^{(0)})|^2}{E_k^{(0)} - E_n^{(0)}} \right] & \text{(linear)} \\ \pm\gamma \frac{4gl_g^2 z_n^2}{15} + \gamma^2 \left[\left(-\frac{1}{2} + \frac{56 z_n^3}{105} \right) \frac{2gl_g^3}{m} + 4g^2 l_g^4 \sum_{k \neq n} a_{nk} \right], & \text{(quadratic)} \end{cases} \tag{41}$$

where α_{nk} is given by (23b). As one can see from (23) and (29), there is a difference between the eigenvalues associated to the constant of motion and those associated to the Hamiltonian. Their relative differences, $\delta E_n = (E_n^H - E_n^K)/E_n^{(0)}$, is given by

$$\frac{\delta E_n}{E_n^{(0)}} = \begin{cases} \alpha^2\left[-\frac{1}{6}\frac{l_g z_n}{6m^2 g} - \frac{4}{9}\frac{l_g^2}{m z_n}\sum_{k\neq n}\frac{|1/2 + mgl_g/(E_k^{(0)} - E_n^{(0)})|^2}{E_k^{(0)} - E_n^{(0)}}\right] & \text{(linear)} \\ \pm\gamma\frac{16gl_g^2 z_n^2}{15}, & \text{(quadratic)} \end{cases} \tag{42}$$

6. CONCLUSION

The classical and quantum problem of a particle bouncing on a hard surface under the influence of gravity and subject to a linear and quadratic velocity dissipative force were treated by using the constant of motion and Hamiltonian. Expression (3) and (11a) gives us the expected damping behavior of the particle on the (x, v) space, but the expression (7) and (11d) shows an unexpected behavior in the (x, p) space (two trajectories on this space, for dissipative parameter one bigger than other, do not follow one under the other all the time). For the quantum case, we have analyzed the eigenvalues for the constant of motion and Hamiltonian up to second order in the dissipation parameter, using perturbation theory. Relation (30) tells us that that there is a difference whether the constant of motion or the Hamiltonian is quantized, and it suggests that one could see this difference experimentally. In this way, one could see whether nature prefers to follow constant of motions rather than Hamiltonians for dissipative systems. Finally, one must observe that for the full linear case (7), it is possible to solve exactly the Shrödinger equation in the momentum representation, and this will be analyzed on a future paper.

APPENDIX

We show here a list of some matrix elements from reference (Gean-Banacloche, 1999; Goodmanson, 2000) and some other calculated from the same reference (an correction of a sign has been made to some matrix elements). Given the functions (20) and $n \neq k$, one has

$$\langle n|k\rangle = \delta_{nk} \tag{43}$$

$$\langle n|z|n\rangle = \frac{2}{3}z_n \qquad \langle n|z|k\rangle = \frac{2(-1)^{n+k+1}}{(z_n - z_k)^2} \tag{44}$$

$$\langle n|z^2|n\rangle = \frac{8}{15}z_n^2 \qquad \langle n|z^2|k\rangle = \frac{24(-1)^{n+k+1}}{(z_n - z_k)^4} \tag{45}$$

$$\langle n|z^3|n\rangle = \frac{3}{7} + \frac{48}{105}z_n^3 \qquad \langle n|z^3|k\rangle = \frac{24(z_n + z_k)(-1)^{n+k+1}}{(z_n - z_k)^4} \tag{46}$$

$$\left\langle n|\frac{d}{dz}|n\right\rangle = 0 \qquad \left\langle n|\frac{d}{dz}|k\right\rangle = \frac{(-1)^{n+k}}{z_n - z_k} \tag{47}$$

$$\left\langle n|\frac{d^2}{dz^2}|n\right\rangle = -\frac{1}{3}z_n \qquad \left\langle n|\frac{d^2}{dz^2}|k\right\rangle = \frac{2(-1)^{n+k}}{(z_n - z_k)^2} \tag{48}$$

$$\left\langle n|\frac{d^3}{dz^3}|n\right\rangle = \frac{1}{2} \qquad \left\langle n|\frac{d^3}{dz^3}|k\right\rangle = \left(\frac{1}{2} + \frac{1}{z_k - z_n}\right)(-1)^{n+k} \tag{49}$$

$$\left\langle n|\frac{d^4}{dz^4}|n\right\rangle = \frac{1}{5}z_n^2 \qquad \left\langle n|\frac{d^4}{dz^4}|k\right\rangle = \frac{-2(z_k - z_n) + 24 - 2z_k(z_k - z_n)^2}{(z_k - z_n)^4}(-1)^{n+k} \tag{50}$$

REFERENCES

Berman, G. P., Borgonovi, F., López, G. V., and Tsifrinovich, V. I. (2003). *Physical Review A* **68**, 012102.

Borges, J. S., Epele, L. N., and Fanchiotti, H. (1988). *Physical Review A* **38**(6), 3101.

Caldeira, A. O. and Leggett, A. T. (1983). *Physica A* **121**, 587.

Dirac, P. A. M. (1992). *The Principles of Quantum Mechanics*, Oxford University Press, Oxford, UK.

Dodonov, V. V. (1981). *Hadron Journal* **4**, 173.

Gean-Banacloche, J. (1999). *American Journal of Physics* **67**(9), 776.

Glauber, R. and Man'ko, V. I. (1984). *Sov. Phys. JEPT* **60**, 450.

Goodmanson, D. M. (2000). *American Journal of Physics* **68**(9), 866.

Huang, B. W., Gu, Z. V., and Qiam, S. W. (1989). *Physical Letters A* **142** (4–5), 203.

Hu, B. L., Paz, J. P., and Zhang, Y. (1992). *Physical Review D* **45**, 2843.

López, G. (1996). *Annals of Physics* **251**(2), 372.

López, G. (1999a). *Revista Mexicana de Fisica* **45**, 1817.

López, G. (1999b). *Partial Differential Equations of First Order and Their Applications to Physics*, World Scientific, Singapore, p. 37.

López, G. (2002). *Revista Mexicana de Fisica* **48**(1), 10.

López, G., and González, G. (2003). *IL Nuovo Cimento B* **118**(2), 107.

López, G., Murgía M., and Sosa, M. (1997). *Modern Physical Letters B* **11**(14), 625.

López, G., Murgía, M., and Sosa, M. (2001). *Modern Physical Letters B* **15**(22), 965.

Mijatovic, M., Veljanoski, B., and Hajdukovic, D. (1984). *Hadronic Journal* **7**(5), 1207.

Negro, F. and Tartaglia, A. (1980a). *Physical Letters A* **77**, 1.

Negro, F. and Tartaglia, A. (1980b). *Physical Review A* **1981**(23), 1591.

Okubo, S. (1981). *Physical Review A* **23**, 2776.

Onofrio, R. and Viola, L. (1996). *Physical Review A* **53**, 3773.

Razavy, M. (1972). *Canadian Journal of Physics* **50**, 2037.

Razavy, M. (1983). *Hadronic Journal* **6** (2), 406.

Stuckeno, C. and Kobe, D. H. (1986). *Physical Review A* **34**(5), 3565.

Unruh, W. G. and Zurek, W. H. (1989). *Physical Review D* **40**, 1071.